Quasi-coherent sheaves

P. Gabriel

Translator's note.

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What follows is a translation of the French seminar talk:

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Contents

p. 1-01

We assume prior knowledge of the definitions and elementary properties of sheaves of modules on a topological space, i.e. [2, chapitre I, §1; chapitre II, §§1–2]. We define a presheaf $\mathscr{P}$ on a base $\mathscr{B}$ of open subsets of a topological space X, with values in a category $\mathcal{C}$, to be the following data:

(a) for every open subset U in $\mathscr{B}$, an object $\mathscr{P}(U)$ of $\mathcal{C}$, that we may also denote by $\Gamma(U,\mathscr{P})$;

(b) for every pair (U,V) of open subsets U in $\mathscr{B}$ such that $U\subset V$, a morphism $\rho_{UV}\colon\mathscr{P}(V)\to\mathscr{P}(U)$. The morphism ρ_{UV} will be called the restriction of V to U. We further suppose that, if $U\subset V\subset W$ are open subsets in $\mathscr{B}$, then $\rho_{UV}\circ\rho_{VW}=\rho_{UW}$.

[*] https://thosgood.com/translations/

The construction of the sheaf $\widetilde{\mathscr{P}}$ associated to a presheaf $\mathscr{P}$ can be easily generalised to the case of presheaves on a base of open subsets. If $\mathfrak{U} = (U_i)_{i\in I}$ is a cover of the open subset $U \in \mathscr{B}$, where $U_i \in \mathscr{B}$, and $U_i \cap U_j \in \mathscr{B}$, and if $\mathscr{P}$ is a presheaf of rings (resp. of modules) on the base $\mathscr{B}$, then we write $\mathrm{H}^0(\mathfrak{U}, \mathscr{P})$ to mean the subring (resp. submodule) of $\prod_{i\in I} \mathscr{P}(U_i)$ given by the $(X_i)_{i\in I}$ such that $\rho_{U_i\cap U_j,U_i}(X_i) = \rho_{U_i\cap U_j,U_j}(X_j)$ for every pair $i, j \in I$. We then have canonical maps:

$$\mathscr{P}(U) \to \mathrm{H}^0(\mathfrak{U}, \mathscr{P}) \to \widetilde{\mathscr{P}}(U);$$

if these maps are injective for every cover $\mathfrak{U}$ satisfying the conditions above, then $\widetilde{\mathscr{P}}(U)$ is the union of the $\mathrm{H}^0(\mathfrak{U}, \mathscr{P})$.

1 Preliminaries on localisation

Let A be a unital commutative ring. A submonoid S of the multiplicative monoid of A (i.e. a non-empty subset of A such that, if it contains s and t, then it contains $s \cdot t$) is said to be ***complete*** if it satisfies the following condition: if $s \cdot t \in S$, then $s \in S$ and $t \in S$. To every multiplicative submonoid S, we associate a complete *p. 1-02* monoid $\widetilde{S}$ in the following way: $s \in \widetilde{S}$ if and only if there exists some t in A such that $s \cdot t \in S$. The prime ideals that meet S also meet $\widetilde{S}$, and vice versa. Furthermore, the complement of $\widetilde{S}$ in A is a union of prime ideals (the ideals that are maximal amongst those that do not meet $\widetilde{S}$ are prime).

If now M denotes a unital A-module, and S a multiplicative submonoid of A, then we denote by M_S the following abelian group:

The set M_S is the quotient of $M \times S$ by the equivalence relation

$$(m, s) = (n, t) \iff \exists r \in S \text{ such that } r(mt - ns) = 0.$$

The addition in $M \times S$ is defined by $(m, s) + (n, t) = (mt + ns, st)$; our equivalence relation is compatible with the addition, whence we get the structure of an abelian group on M_S. We denote by m/s the class of (m, s) in M_S.

We have a bilinear map $A_S \otimes_{\mathbb{Z}} M_S \to M_S$ defined by passing to quotients from the map $((a, s), (m, t)) \mapsto (am, st)$. Taking $M = A$, we see that A_S is a ring and, more generally, M_S is an A_S-module. The map $\psi\colon m \mapsto m/1$ from M to M_S is compatible with the ring homomorphism $\varphi\colon a \mapsto a/1$ from A to A_S (i.e. it is a homomorphism of abelian groups such that $\psi(a \cdot m) = \varphi(a) \cdot \psi(m)$).

We can furthermore easily prove the following claims: the correspondence $M \mapsto M_S$ is functorial (in the evident way); the functor $M \mapsto M_S$ is exact (i.e. if $0 \to M \to M' \to M'' \to 0$ is an exact sequence of A-modules, then $0 \to M_S \to M'_S \to M''_S \to 0$ is an exact sequence of A-modules) and commutes with inductive limits (i.e. if $(M_i)_{i\in I}$ is an inductive system of A-modules, and L an inductive limit of this system, then the $(M_i)_S$ form an inductive system of A_S-modules, and the homomorphisms $(M_i)_S \to L_S$ induced by the $M_i \to L$ define L_S as an inductive limit of the system $((M_i)_S)$); the canonical map $M \otimes_A A_S \to M_S$ is bijective.

If S and T are submonoids, then the abelian groups $(M_S)_T$, $(M_T)_S$, and $M_{S\cdot T}$ are all canonically isomorphic to one another, where $S \cdot T$ denotes the monoid given by products $s \cdot t$ with $s \in S$ and $t \in T$. Identifying the rings $(A_S)_T$, $(A_T)_S$, and $A_{S \cdot T}$, which are themselves all canonically isomorphic to one another, the isomorphisms between $(M_S)_T$, $(M_T)_S$, and $M_{S \cdot T}$ are isomorphisms of modules. p. 1-03

If S is contained in a submonoid S' of the multiplicative monoid of A, then there is a canonical map from M_S to $M_{S'}$ that sends each element m_s of M_S (where $m \in M$ and $s \in S$) to the element of $M_{S'}$ denoted by the same symbol. The map $A_S \to A_{S'}$ is a ring homomorphism; the map $M_S \to M_{S'}$ is that which corresponds to the map $M \otimes_A A_S \to M \otimes_A A_{S'}$, induced by $A_S \to A_{S'}$. If $S' = \widetilde{S}$ is the smallest complete monoid containing S, then the map $M_S \to M_{\widetilde{S}}$ is bijective. Finally, if S is the union of an increasingly-ordered filtered family of monoids S_i, then M_S can be identified with $\varinjlim M_{S_i}$.

2 The prime spectrum of a commutative ring

Let A be a unital commutative ring, and $V(A)$ the set of prime ideals of A. If $\mathfrak{a}$ is an ideal of A, then we denote by $W(\mathfrak{a})$ the set of prime ideals that contain $\mathfrak{a}$, and $U(\mathfrak{a}) = V(A) \setminus W(\mathfrak{a})$.

Then

$$
\begin{gathered}
W(\mathfrak{a}) \subset V(A), \quad U(\mathfrak{a}) \subset V(A), \\
W(\textstyle\sum \mathfrak{a}_i) = \displaystyle\bigcap_i W(\mathfrak{a}_i), \quad W(\mathfrak{a} \cap \mathfrak{b}) = W(\mathfrak{a} \cdot \mathfrak{b}) = W(\mathfrak{a}) \cup W(\mathfrak{b}), \\
U(\textstyle\sum \mathfrak{a}_i) = \displaystyle\bigcup_i U(\mathfrak{a}_i), \quad U(\mathfrak{a} \cap \mathfrak{b}) = U(\mathfrak{a} \cdot \mathfrak{b}) = U(\mathfrak{a}) \cap U(\mathfrak{b}).
\end{gathered}
$$

The set $U(\mathfrak{a})$ increases with $\mathfrak{a}$ and $W(\mathfrak{a})$ decreases when $\mathfrak{a}$ increases. Finally, $W(\mathfrak{a}) = W(\mathfrak{b})$ if and only if every element of $\mathfrak{a}$ has a power in $\mathfrak{b}$, and vice versa: indeed, to say that $W(\mathfrak{a}) \subset W(\mathfrak{b})$ is to say that every prime ideal that contains $\mathfrak{a}$ also contains $\mathfrak{b}$, i.e. that $\mathfrak{b}$ is contained in the intersection of the prime ideals containing $\mathfrak{a}$, and it is classical that this latter intersection consists of the elements that have a power in $\mathfrak{a}$.

The sets $U(\mathfrak{a})$ are the open subsets of a topology on $V(A)$, and, endowed with this topology, $V(A)$ is called the ***prime spectrum of*** A.

If the ideal $\mathfrak{a}$ is generated by the (f_i), then $U(\mathfrak{a})$ is the union of the $U((f_i)) = U_{f_i}$; since $U_f \cap U_g = U_{fg}$, the U_f thus form a base of open subsets when f runs over the elements of A. Every open subset of the form U_f is said to be ***special***. Every special open subset is ***quasi-compact***: indeed, if U_f is the union of some $U(\mathfrak{a}_i)$, then $U_f = \bigcup_i U(\mathfrak{a}_i) = U(\sum \mathfrak{a}_i)$, and a power f^n of f belongs to $\sum_i \mathfrak{a}_i$, and thus to the sum of a finite number of the $\mathfrak{a}_i$.

In particular, $V = U_i$ is quasi-compact. Finally, if the ring A is ***Noetherian***, then every increasing sequence of ideals stabilises, and the same is true for every increasing sequence of open subsets. The prime spectrum is thus a ***Zariski topological space***. p. 1-04

3 Quasi-coherent sheaves on $V(A)$

Now let M be a unital module over the ring A. We are going to define a ***presheaf*** $\mathfrak{M}^*$ ***on the base of special open subsets of*** $V(A)$. For this, we will associate, to each $f \in A$, the multiplicatively-stable system S_f given by the complement in A of the union of the prime ideals of U_f. The system S_f is the complete multiplicatively-stable system generated by f. We then have the equivalence:

$$S_f \subset S_g \mapsto U_f \supset U_g.$$

With this in mind, we will define the presheaf $\mathfrak{M}^*$ by the formulas $\mathfrak{M}^*(U_f) = M_{S_f}$, which we will also denote by $\mathfrak{M}_f$; if $U_f \supset U_g$, then the restriction map $\rho\colon \mathfrak{M}^*(U_f) \to \mathfrak{M}^*(U_g)$ is the canonical map from $\mathfrak{M}_{S_f}$ to $\mathfrak{M}_{S_g}$. These definitions clearly do not depend on the elements f and g that define U_f and U_g; furthermore, the axioms of a presheaf are satisfied, thanks to the properties of localisation.

We will denote by $\mathfrak{M}$ the sheaf associated to the presheaf $\mathfrak{M}^*$. Since the A_f are rings, and the M_f are A_f-modules, $\mathfrak{A}^*$ is a presheaf of rings, $\mathfrak{M}^*$ a presheaf of $\mathfrak{A}^*$-modules, $\mathfrak{A}$ a sheaf of rings, and $\mathfrak{M}$ a sheaf of $\mathfrak{A}$-modules. Furthermore, the functors that send M to $\mathfrak{M}^*$ and $\mathfrak{M}$ are exact, since the functor that sends M to the M_f is exact. We define an *algebraic sheaf* on $V(A)$ to be any sheaf of modules over the sheaf of rings $\mathfrak{A}$. We define a *quasi-coherent sheaf* on $V(A)$ to be any sheaf isomorphic to a sheaf of the form $\mathfrak{A}(M)$ for some M. We will first be interested in the sections of such a sheaf over a special open subset U_f.

For this, we note that the prime ideals of A_f are the images under the map $A \mapsto A_f$ of the prime ideals of A that do not contain f. Furthermore, the canonical application thus defined from the spectrum $V(A_f)$ of A_f to $V(A)$ is a homomorphism from $V(A_f)$ to U_f. Finally, the sheaf on $V(A_f)$ associated to the module M_f can be identified with the restriction of $\mathfrak{M}$ to U_f.

More generally, if S is a multiplicatively-stable system (that we can assume to be complete) of A, then let E_S be the topological subspace of $V(A)$ consisting of the prime ideals that do not meet S. The canonical map $A \mapsto A_S$ induces a homomorphism from $V(A_S)$ to E_S; if S is generated by the f_i, then E_S is the *p. 1-05*
intersection of the U_{f_i}, and, conversely, every intersection of special open subsets of $V(A)$ is of the form E_S. We will see that the restriction of $\mathfrak{M}$ to the subspace E_S can be identified with the sheaf $\mathfrak{M}_S$ associated to the A_S-module M_S.

For every prime ideal $\mathfrak{p}$, we denote by $A_\mathfrak{p}$ and $M_\mathfrak{p}$ the ring and the module obtained by localisation of the multiplicative system $S = A \setminus \mathfrak{p}$.

Theorem 1. *(a) If $\mathfrak{p}$ is a point of $V(A)$, then the localised ring $\mathfrak{A}_\mathfrak{p}$ (the fibre of the sheaf $\mathfrak{A}$ over $\mathfrak{p}$) is canonically isomorphic to $A_\mathfrak{p}$; Identifying $\mathfrak{A}_\mathfrak{p}$ with $A_\mathfrak{p}$, the localised module $\mathfrak{M}_\mathfrak{p}$ of the sheaf $\mathfrak{M}$ at $\mathfrak{p}$ is canonically isomorphic to $M_\mathfrak{p}$, and thus to $M \otimes_A A_\mathfrak{p}$.*

(b) The canonical map from $M_f = \mathfrak{M}^*(U_f)$ to $\mathfrak{M}(U_f)$ is an isomorphism.

Proof. The first claim of (a) follows from the fact that

$$\mathfrak{A}_\mathfrak{p} = \varinjlim_{U_f \ni \mathfrak{p}} \mathfrak{A}^*(U_f) = \varinjlim_{f \notin \mathfrak{p}} A_f = A_\mathfrak{p};$$

the second claim of (a) can be proven in the same way.

We now show that the map $M_f \to \mathfrak{M}(U_f)$ is *injective*: Since $U_f = V(A_f)$, we can always assume that $f = 1$ and $U_f = V(A)$. The kernel of the map $M \to \mathfrak{M}(V)$ then consists of the m such that $m \otimes_A 1_{A_\mathfrak{p}} = 0$ for every prime ideal $\mathfrak{p}$, i.e. the m whose annihilator is not contained in any prime ideal: the kernel is thus zero.

Similarly, the map $M_f \to \mathfrak{M}(U_f)$ is *surjective*: it suffices to prove this in the case where $f = 1$ and $U_f = V$. By the above, $\mathfrak{M}(V) = \Gamma(\mathfrak{M})$ is the union of the $\mathrm{H}^0(\mathfrak{U}, \mathfrak{M}^*)$ where $\mathfrak{U}$ runs over the finite covers of V by special open subsets; it thus suffices to prove that the map $M \to \mathrm{H}^0(\mathfrak{U}, \mathfrak{M})$ is surjective for every finite cover by special open subsets.

For this, let $(f_i)_{i=1,\ldots,p}$ be elements of A such that $V = \bigcup U_{f_i}$, and let X_i be the elements of M_{f_i} such that X_i and X_j have the same image in $M_{f_i \cdot f_j}$; even if we replace f_i by one of its powers, we can still suppose that X_i is of the form m_i / f_i, where $m_i \in M$. Since X_i and X_j have the same image in $M_{f_i \cdot f_j}$,

$$\frac{m_i f_j - m_j f_i}{f_i f_j} = 0 \in M_{f_i \cdot f_j}.$$

In other words, p. 1-06

$$(f_i \cdot f_j)^r (m_i f_j \cdot m_j f_i) = 0 \in M \text{ for } r \text{ large enough.}$$

Also, the $U_{f_i} = U_{f_i^{r+1}}$ cover V, i.e. the f_i^{r+1} generate A, i.e. we have a relation of the form $1 = \sum_{i=1}^p a_i f_i^{r+1}$, where $a_i \in A$. It follows that, if $m = \sum_i m_i a_i f_i^{r+1}$, then

$$f_j^{r+1} = \sum_i a_i m_i f_i^r f_j^{r+1} = \sum_i a_i m_j f_j^r f_i^{r+1} = m_j f_j^r$$

and, in M_{f_j}, we have the equality $X_j = m_j / f_j = m/1$ for all j. □

Corollary 1. *If S is a multiplicatively-stable subset of A, then the restriction of the sheaf $\mathfrak{M}$ to E_S can be identified with the sheaf $\mathfrak{M}_S$ on $V(A_S)$ associated to M_S.*

Proof. Let f be an element of A. The open subset $E_S \cap U_f$ of E_S can then be identified with $E_{S \cdot S_f}$, and, consequently,

$$\mathfrak{M}_S(E_S \cap U_f) = M_{S \cdot S_f} = (M_S)_f = \varinjlim_{U_g \supset E_S} M_{f \cdot g}.$$

The natural maps $M_{f \cdot g} = \mathfrak{M}(U_{fg}) \to \mathfrak{M}(E_S \cap U_f)$ thus extend to give a map

$$\mathfrak{M}_S(E_S \cap U_f) \to \mathfrak{M}(E_S \cap U_f)$$

and induce a morphism of sheaves:

$$\mathfrak{M}_S \to \mathfrak{M} | E_S.$$

But if $\mathfrak{p} \in E_S$, then the fibre of $\mathfrak{M}_S$ over $\mathfrak{p}$ is exactly $(M_S)_\mathfrak{p} = M_\mathfrak{p}$, and the above map induces an isomorphism over each $\mathfrak{p}$: the map is thus an isomorphism of sheaves. □

p. 1-07

Corollary 2. *M_S is the module of sections of $\mathfrak{M}$ over E_S.*

Corollary 3. *If M and N are A-modules, then the map*

$$\varphi\colon \operatorname{Hom}(M,N) \to \operatorname{Hom}(\mathfrak{M},\mathfrak{N})$$

defined by the functor $M \mapsto \mathfrak{M}$ is bijective.

Proof. We will exhibit an inverse map ψ. For this, let Γ be the functor that sends any sheaf of $\mathfrak{A}$-modules to its module of sections over $V(A)$. The functor Γ defines a map from $\operatorname{Hom}(\mathfrak{M},\mathfrak{N})$ to $\operatorname{Hom}(\Gamma(\mathfrak{M}),\Gamma(\mathfrak{N}))$, and thus, since this latter group is isomorphic to $\operatorname{Hom}(M,N)$ by an explicit isomorphism described above, a map ψ from $\operatorname{Hom}(\mathfrak{M},\mathfrak{N})$ to $\operatorname{Hom}(M,N)$. Then φ and ψ are inverse to one another. □

It follows from this corollary, along with the exactness of the functor $M \mapsto \mathfrak{M}$, that, if $u\colon \mathscr{F} \to \mathscr{G}$ is a morphism of quasi-coherent sheaves, then $\operatorname{Ker} u$ and $\operatorname{Coker} u$ are quasi-coherent sheaves.

Theorem 2. *If $\mathscr{F}$ is an algebraic sheaf on $V(A)$, then the following are equivalent:*

(a) $\mathscr{F}$ is quasi-coherent.

(b) For every special open subset U_f, the natural map

$$\Gamma(V,\mathscr{F}) \otimes_A A_f \to \Gamma(U_f,\mathscr{F})$$

is bijective.

(c) For every point $\mathfrak{p}$ of V, the natural map

$$\Gamma(V,\mathscr{F}) \otimes_A A_\mathfrak{p} \to \mathscr{F}_\mathfrak{p}$$

is bijective.

(d) $\mathscr{F}$ is locally isomorphic to a quasi-coherent sheaf, i.e. every point $\mathfrak{p}$ of V has a special neighbourhood U_f such that $\mathscr{F}|U_f$ is a coherent sheaf.

Proof. We have already seen that (a) $\implies$ (b), (a) $\implies$ (c), and (a) $\implies$ (d).

We will first show that (b) $\implies$ (a): Let $M = \Gamma(V,\mathscr{F})$. Then, for every affine open subset U_f we have restrictions: $M \to \Gamma(U_f,\mathscr{F})$, and since $\Gamma(U_f,\mathscr{F})$ is an A_f-module, it follows that we have a map: p. 1-08

$$M_f = M \otimes_A A_f \to \Gamma(U_f,\mathscr{F}).$$

These maps induce a morphism of sheaves: $\mathfrak{M} \to \mathscr{F}$.

This morphism will be bijective if the induced maps $\mathfrak{M}(U_f) \to \mathscr{F}(U_f)$ are bijective (and so (b) $\implies$ (a)) or if the induced maps $\mathfrak{M}_\mathfrak{p} \to \mathscr{F}_\mathfrak{p}$ are bijective (and so (c) $\implies$ (a)).

It remains only to show that (d) $\implies$ (a). The proof that follows is due to Grothendieck.

We note first of all that the map $A \to A_f$ endows every A_f-module with an A-module structure, and thus associates, to every quasi-coherent sheaf $\mathscr{F}$ on U_f, a quasi-coherent sheaf $\overline{\mathscr{F}}$ on V (we will return later to this operation). Furthermore, if U_g is an affine open subset of V, then:

$$\Gamma(U_g, \overline{\mathscr{F}}) = \Gamma(V, \overline{\mathscr{F}})_g = \Gamma(U_f, \mathscr{F})_g = \Gamma(U_f \cap U_g, \mathscr{F}).$$

So take some algebraic sheaf $\mathscr{G}$ on V such that there exists a cover of V by special open subsets U_{f_i} with the condition that $\mathscr{G}|U_{f_i}$ is quasi-coherent. We will now show that $\mathscr{G}$ is quasi-coherent.

Since $\mathscr{G}$ is a sheaf, we have, for all U_g, an exact sequence of the form:

$$0 \to \Gamma(U_g, \mathscr{G}) \to \prod_i \Gamma(U_g \cap U_i, \mathscr{G}) \to \prod_{i<j} \Gamma(U_g \cap U_i \cap U_j, \mathscr{G})$$

or even

$$0 \to \Gamma(U_g, \mathscr{G}) \to \prod_i \Gamma) U_g, \overline{\mathscr{G}|U_i}) \to \prod_{i \leqslant j} \Gamma(U_g, \overline{\mathscr{G}|U_i \cap U_j}).$$

The two latter terms of the exact sequence are the sections over U_g of the sheaves associated to the modules

$$M = \prod_i \Gamma(U_i, \mathscr{G}|U_i) \text{ and } N = \prod_{i<j} \Gamma(U_i \cap U_j, \mathscr{G}).$$

Letting g vary, we obtain, by taking the limit, an "exact sequence of sheaves": $0 \to \mathscr{G} \to \mathfrak{M} \to \mathfrak{N}$. The sheaf $\mathscr{G}$ is thus the kernel of a morphism of quasi-coherent sheaves: $\mathscr{G}$ is quasi-coherent. □

4 Coherent sheaves on $V(A)$

p. 1-09

From now on we will suppose that A ***is a Noetherian ring***. Then it is well known that the category of Noetherian A-modules agrees with that of A-modules of finite type. We define a ***coherent sheaf on*** $V(A)$ to be any sheaf $\mathscr{F}$ that is isomorphic to a sheaf of the type $\mathfrak{M}$, where M is an A-module of finite type. The sheaf $\mathscr{F}$ is thus quasi-coherent, and it is a sheaf of finite type (a sheaf $\mathscr{F}$ of $\mathfrak{A}$-modules on a topological space X is said to be ***of finite type*** if every point of X admits a neighbourhood U along with a surjection $\mathfrak{A}^p|U \to \mathscr{F}|U \to 0$, where $\mathfrak{A}^p$ is sheaf given by the direct sum of p sheaves, each isomorphic to $\mathfrak{A}$. Such a surjection induces a map

$$\Gamma(U, \mathfrak{A}^p) = \Gamma(U, \mathfrak{A})^p \to \Gamma(U, \mathscr{F})$$

and defines p surjections from $\mathscr{F}$ to U, i.e. the images of the basis vectors of $\Gamma(U, \mathfrak{A})^p$. Conversely, the data of p sections of $\mathscr{F}$ over U defines a map $A^p|U \to \mathscr{F}|U$).

Theorem 3. *The following are equivalent (if A is Noetherian and if $\mathscr{F}$ is an algebraic sheaf on $V(A)$):*

(a) $\mathscr{F}$ is a coherent algebraic sheaf on $V(A)$.

(b) $\mathscr{F}$ is quasi-coherent and of finite type.

(c) $\mathscr{F}$ is of finite type, and, for every open subset U and every morphism $\varphi\colon \mathscr{G} \to \mathscr{F}|U$, where $\mathscr{G}$ is a sheaf of finite type on U, the kernel $\operatorname{Ker}\varphi$ is of finite type on U.

Proof. We already know that (a) $\implies$ (b). Conversely, if $\mathscr{F}$ is quasi-coherent and of finite type, then there exists a cover $(U_i)_{i\in I}$ of $V(A)$ by special open subsets such that $\Gamma(U_{f_i}, \mathscr{F})$ is an A_{f_i}-module of finite type. Since $V(A)$ is quasi-compact, we can always assume that this cover is finite, and that $\mathscr{F}$ is of the form $\mathfrak{M}$: we then need to prove that M is an A-module of finite type. But, for all i, there exists a finite number of $m_{i_k} \in M$ that generate M_{f_i}, and, if N denotes the submodule generated by all the m_{i_k}, then N is an A-module of finite type, and clearly $\mathfrak{N} = \mathfrak{M}$, whence $N = M$.

Now for (c) $\implies$ (b): If $\mathscr{F}$ is of finite type, then every point admits a special neighbourhood U over which we have a surjection: p. 1-10

$$\mathfrak{A}^p|U \to \mathscr{F}|U \to 0.$$

Since the kernel of this surjection is also of finite type, we in fact have an exact sequence (provided that the special open subset U is small enough):

$$\mathfrak{A}^q|U \to \mathfrak{A}^p|U \to \mathscr{F}|U \to 0.$$

The sheaf $\mathscr{F}$ is thus, on every small enough special open subset, the cokernel of a morphism of quasi-coherent sheaves: so $\mathscr{F}$ is locally quasi-coherent, and is thus quasi-coherent.

Finally, (a) $\implies$ (c): If $\mathscr{F}$ is coherent, then $\mathscr{F}$ is of finite type. Furthermore, if $\varphi\colon \mathscr{G} \to \mathscr{F}|U$ is a morphism, then we can always assume that U is a special open subset that is small enough such that we have a surjection $\psi\colon \mathfrak{A}^p|U \to \mathscr{G} \to 0$.

We then have the following diagram:

$$\begin{array}{ccccc}
 & & \mathfrak{A}^p|U & & \\
 & & \downarrow{\scriptstyle\psi} & \searrow^{\chi} & \\
\operatorname{Ker}\varphi & \longrightarrow & \mathscr{G} & \xrightarrow[\varphi]{} & \mathscr{F}|U \\
 & & \downarrow & & \\
 & & 0 & &
\end{array}$$

where $\chi = \varphi \circ \psi$.

The sheaves $\mathfrak{A}^p|U$ and $\mathscr{F}|U$ are coherent on U, and χ is thus induced by a morphism $\chi'\colon \mathfrak{A}^p(U) \to \mathscr{F}(U)$. This "resolution" of $\Gamma(U,\mathscr{F}) = \mathscr{F}(U)$ extends to a resolution

$$\Gamma(U,A)^q \to \Gamma(U,A)^p \to \Gamma(U,\mathscr{F}).$$

This latter exact sequence allows us to complete the diagram and show that, on U, $\operatorname{Ker}\varphi$ is the quotient of a sheaf of finite type:

$$\begin{array}{ccccc}
\mathfrak{A}^q|U & \longrightarrow & \mathfrak{A}^p|U & \xrightarrow{\ \chi\ } & \mathscr{F}|U \\
\downarrow & & \psi\downarrow & & \updownarrow \\
\operatorname{Ker}\varphi & \longrightarrow & \mathscr{G} & \xrightarrow[\varphi]{} & \mathscr{F}|U \\
\downarrow & & \downarrow & & \\
0 & & 0 & &
\end{array}$$

□

p. 1-11

Remark. Claim (c) gives a characterisation of coherent sheaves, which does not require the base of special open subsets of $V(A)$. In fact, if V is an arbitrary topological space, and $\mathfrak{A}$ a sheaf of rings on V, then the sheaves of $\mathfrak{A}$-modules that satisfy (c) are the objects of an abelian subcategory of the category of sheaves of $\mathfrak{A}$-modules (see Serre [4]). In the case considered here, the quasi-coherent sheaves are the inductive limits of coherent sheaves.

5 The maximal spectrum of a Jacobson ring

We are going to apply the above to algebraic geometry. For this, we denote by $\Omega(A)$ the *maximal spectrum of* A, which is defined to be the topological subspace of $V(A)$ consisting of the maximal ideals of A. We further suppose that A is a *Jacobson ring*, i.e. that every prime ideal is the intersection of maximal ideals (see [1] and [3]). The following proposition then holds true:

Proposition. *The following are equivalent:*

(a) A *is a Jacobson ring.*

(b) The correspondence $U \mapsto U \cap \Omega(A)$ *between open subsets of* $V(A)$ *and of* $\Omega(A)$ *is bijective.*

(c) The correspondence $W \mapsto W \cap \Omega(A)$ *between closed subsets of* $V(A)$ *and of* $\Omega(A)$ *is bijective.*

Proof. The equivalence of (b) and (c) is trivial. On the other hand, the closed subsets of V correspond to the ideals of A that are intersections of prime ideals. The closed subsets of Ω correspond to the ideals of A that are intersections of maximal ideals. The two families of ideals agree if and only if A is a Jacobson ring. □

Under this last hypothesis, the spaces $V(A)$ and $\Omega(A)$ thus have the same lattice of open subsets. It clearly follows that the sheaves on V and on Ω "are in bijective correspondence". In particular, we define a *special open subset of* Ω to be the restriction of any special open subset of V, and a *quasi-coherent* (resp. *coherent*) *algebraic sheaf on* Ω to be the restriction of any quasi-coherent (resp. coherent) algebraic sheaf on V. The properties of these sheaves on V extend (up to evident modifications) to their restrictions to Ω.

In particular, every algebra of finite type over a field is a Jacobson ring. If k is an algebraically closed field, V an affine algebraic set over k, and A its coordinate ring, then V is homeomorphic to the maximal spectrum $\Omega(A)$ of A. The sheaf *p. 1-12*
of rings $\mathfrak{A}|\Omega(A)$ is called the *sheaf of germs of regular functions on* V.

Similarly, a sheaf of modules over this sheaf of rings is said to be *algebraic* (resp. *quasi-coherent algebraic*, resp. *coherent algebraic*) if it is the restriction of a sheaf on $V(A)$ with the named property.

6 Quasi-coherent sheaves on an algebraic variety

More generally, if X is an arbitrary algebraic set over k, then we define a *quasi-coherent* (resp. *coherent*) *algebraic sheaf on* X to be any sheaf $\mathscr{F}$ such that there exists a cover of X by affine open subsets U_i such that $\mathscr{F}|U_i$ is quasi-coherent (resp. coherent). By the remark that follows Theorem 3, this is equivalent to saying that, for every affine open subset U, the sheaf $\mathscr{F}|U$ is quasi-coherent (resp. coherent).

In particular, the sheaf of rings $\mathfrak{A}$ such that, for every affine open subset U, $\mathfrak{A}(U)$ is the ring of regular functions on U (with the evident restrictions), is a coherent sheaf. We call it the *sheaf of germs of regular functions.* Every sheaf of modules over $\mathfrak{A}$ is said to be *algebraic.*

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