On the representation of definite forms as the sum of squares of forms

David Hilbert

Translator's note.

This text is one of a series[*] of translations of various papers into English. The translator takes full responsibility for any errors introduced in the passage from one language to another, and claims no rights to any of the mathematical content herein.

What follows is a translation of the German paper:

HILBERT, D. "Ueber die Darstellung definiter Formen als Summe von Formenquadraten". *Mathematische Annalen*, Volume **32** (1888), 342–350. http://eudml.org/doc/157385

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An algebraic form of even order n with real coefficients and m homogeneous variables is said to be *definite* if it takes a positive value for every system of real values of the m variables and, moreover, has a non-zero discriminant. A form with real coefficients is also called a *real form* for short.

It is known that *any definite quadratic form* with m variables can be expressed as the sum of m squares of real linear forms. In the same way, *any definite binary form* can be represented as the sum of the squares of two real forms, as can be seen by a suitable factor decomposition of the form. Since the representation in question reveals the definite character of the form in the simplest possible way, an investigation of the possibility of such a representation in general seems to be of interest. For the case where $n = 4$ and $m = 3$, we have the following theorem:

Every definite ternary quartic form can be written as the sum of the squares of three quadratic forms.

To prove this, consider a ternary quartic form F that can be written as the sum of the squares of three quadratic forms φ, ψ, and χ. If the form F can also be written as the sum of the squares of three quadratic forms $\varphi + \varepsilon\varphi'$, $\psi + \varepsilon\psi'$, and $\chi + \varepsilon\chi'$, where ε is an infinitely small constant, then the comparison of the two expressions leads to the relation

$$\varphi\varphi' + \psi\psi' + \chi\chi' = 0. \tag{1}$$

The three equations

$$\varphi = 0, \quad \psi = 0, \quad \chi = 0 \tag{2}$$

cannot have a common solution. By (1), the four common solutions of the latter two equations must make the quadratic form φ' vanish; thus *p. 343*

[*] https://thosgood.com/translations/

$$\varphi' = \alpha\psi + \gamma\chi,$$

and, similarly,

$$\psi' = \beta\varphi + \zeta\chi,$$
$$\chi' = \delta\varphi + \vartheta\psi.$$

By substituting these equations into (1), we obtain the following relations for the introduced constants:

$$\alpha + \beta = 0, \quad \gamma + \delta = 0, \quad \zeta + \vartheta = 0.$$

As soon as the resultant of the three equations (2) is non-zero, the identity (1) can only be satisfied by a linear combination of the three solutions:

$$\left\{\begin{array}{rcr} \varphi' & = & 0 \\ \psi' & = & \chi \\ \chi' & = & -\psi \end{array}\right\} \quad \left\{\begin{array}{rcr} \varphi' & = & -\chi \\ \psi' & = & 0 \\ \chi' & = & \varphi \end{array}\right\} \quad \left\{\begin{array}{rcr} \varphi' & = & \psi \\ \psi' & = & -\varphi \\ \chi' & = & 0 \end{array}\right\}$$

i.e. all possible solution systems represent the same form F, and so all representations of F as the sum of the squares of three forms consist only of φ, ψ, and χ. Since the system of these three quadratic forms has 18 coefficients, whereas the quartic form F has 15 coefficients, it follows from the above remarks that every ternary quartic form can be expressed as the sum of the squares of three forms. *

The coefficients of the three forms φ, ψ, χ have three more arbitrary parameters, and therefore only take specific values when we impose any three mutually dependent conditions on them. If we impose such conditions, then there is only a finite number of systems of forms φ, ψ, χ that can give a representation of the quartic form F as a sum of squares. If, with any choice of such conditions, two systems of forms **TODO**: **approach one another**, then, by the above, it must be the case that the resultant of the forms φ, ψ, χ in question, and thus the discriminant of the form F which is to be represented, disappear. The singular case thus described is of importance for the additional conclusion.

Namely, let F, F', F'' be three definite ternary quartic forms, and p, p', p'' three positive variables, whose relations to one another may be represented by points inside a coordinate triangle. We then take all the points p, p', p'' such that the equation *p. 344*

$$pF + p'F' + p''F'' = 0 \tag{3}$$

defines a quartic curve with two or more double points.

*The general principle on which this is based comes from L. Kronecker, c.f. *Mathematische Annalen* Bd. **13**, p. 549.