Regular singular differential equations

Pierre Deligne

Translator's note.

This text is one of a series[*] of translations of various papers into English. The translator takes full responsibility for any errors introduced in the passage from one language to another, and claims no rights to any of the mathematical content herein.

What follows is a translation of the French book:

DELIGNE, P. *Equations Différentielles à Points Singuliers Réguliers.* Springer–Verlag, Lecture Notes in Mathematics **163** (1970). `https://publications.ias.edu/node/355`

We have also made changes following the errata, which was written in April 1971, by P. Deligne, at Warwick University.

[*] `https://thosgood.com/translations/`

Contents

Chapter 0

Introduction

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If X is a (non-singular) complex-analytic manifold, then there is an equivalence between the notions of

a) local systems of complex vectors on X; and

b) vector bundles on X endowed with an integrable connection.

The latter of these two notions can be adapted in an evident way to the case where X is a non-singular algebraic variety over a field k (which we will take here to be of characteristic 0). However, general algebraic vector bundles with integrable connections are pathological (see ??); we only obtain a reasonable theory if we impose a "regularity" condition at infinity. By a theorem of Griffiths [?], this condition is automatically satisfied for "Gauss-Manin connections" (see ??). In dimension one, this is closely linked to the idea of regular singular points of a differential equation (see ?? and ??).

In Chapter I, we explain the different forms that the notion of an integrable connection can take. In Chapter II, we prove the fundamental facts concerning regular connections. In Chapter III, we translate certain results that we have obtained into the language of Nilsson class functions, and, as an application of the regularity theorem (??), we explain the proof by Brieskorn [?] of the monodromy theorem.

These notes came from the non-crystalline part of a seminar given at Harvard during the autumn of 1969, under the title: "Regular singular differential equations and crystalline cohomology".

I thank the assistants of this seminar, who had to be subjected to often unclear exposés, and who allowed me to bring numerous simplifications.

I also thank N. Katz, with whom I had numerous and useful conversations, and to whom are due the principal results of section ??.

Notation and terminology

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Within a single chapter, the references follow the decimal system. A reference to a different chapter (resp. to the current introduction) is preceded by the Roman numeral of the chapter (resp. by 0).

We will use the following definitions:

(0.1) *analytic space*: the analytic spaces are complex and of locally-finite dimension. They are assumed to be σ-compact, but not necessarily separated.

(0.2) *multiform function*: a synonym for multivalued function — for a precise definition, see **??**.

(0.3) *immersion*: following the tradition of algebraic geometers, immersion is a synonym for "embedding".

(0.4) *smooth*: a morphism $f\colon X \to S$ of analytic spaces is smooth if, locally on X, it is isomorphic to the projection from $D^n \times S$ to S, where D^n is an open polydisc.

(0.5) *locally paracompact*: a topological space is locally paracompact if every point has a paracompact neighbourhood (and thus a fundamental system of paracompact neighbourhoods).

(0.6) non-singular (or smooth) *complex algebraic variety*: a smooth scheme of finite type over $\mathrm{Spec}(\mathbb{C})$.

(0.7) (complex) *analytic manifold*: a non-singular (or smooth) analytic space.

(0.8) *covering*: following the tradition of topologists, a covering is a continuous map $f\colon X \to Y$ such that every point $y \in Y$ has a neighbourhood V such that $f|V$ is isomorphic to the projection from $F \times V$ to V, where F is discrete.

Chapter I

Dictionary

In this chapter, we explain the relations between various aspects and various uses of the notion of "local systems of complex vectors". The equivalence between the points of view considered has been well known for a long time. *p. 3*

We do not consider the "crystalline" point of view; see [?, ?].

I.1 Local systems and the fundamental group

Definition 1.1. Let X be a topological space. A *complex local system* on X is a sheaf of complex vectors on X that, locally on X, is isomorphic to a constant sheaf $\mathbb{C}^n$ (n$\in \mathbb{N}$).

1.2. Let X be a locally path-connected and locally simply path-connected topological space, along with a basepoint $x_0 \in X$. To avoid any ambiguity, we point out that:

a) The fundamental group $\pi_1(X, x_0)$ of X at x_0 has elements given by homotopy classes of loops based at x_0;

b) If $\alpha, \beta \in \pi_1(X, x_0)$ are represented by loops a and b, then $\alpha\beta$ is represented by the loop ab obtained by juxtaposing b and a, in that order.

Let $\mathscr{F}$ be a locally constant sheaf on X. For every path $a\colon [0,1] \to X$, the inverse image $a^*\mathscr{F}$ of $\mathscr{F}$ on $[0,1]$ is a locally constant, and thus constant, sheaf, and there exists exactly one isomorphism between $a^*\mathscr{F}$ and the constant sheaf defined by the set $(a^*\mathscr{F})_0 = \mathscr{F}_{a(0)}$. This isomorphism defines an isomorphism $a(\mathscr{F})$ between $(a^*\mathscr{F})_0$ and $(a^*\mathscr{F})_1$, i.e. an isomorphism

$$a(\mathscr{F})\colon \mathscr{F}_{a(0)} \to \mathscr{F}_{a(1)}.$$

This isomorphism depends only on the homotopy class of a, and satisfies $ab(\mathscr{F}) = a(\mathscr{F}) \cdot b(\mathscr{F})$. In particular, $\pi_1(X, x_0)$ acts (on the left) on the fibre $\mathscr{F}_{x_0}$ of $\mathscr{F}$ at x_0. It is well known that:

Proposition 1.3. *Under the hypotheses of (1.2), with X connected, the functor $\mathscr{F} \mapsto \mathscr{F}_{x_0}$ is an equivalence between the category of locally constant sheaves on X and the category of sets endowed with an action by the group $\pi_1(X, x_0)$.*

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Corollary 1.4. *Under the hypotheses of (1.2), with X connected, the functor $\mathscr{F} \mapsto \mathscr{F}_{x_0}$ is an equivalence between the category of complex local systems on X and the category of complex finite-dimensional representations of $\pi_1(X, x_0)$.*

1.5. Under the hypotheses of (1.2), if $a\colon [0,1] \to X$ is a path, and b a loop based at $a(0)$, then $aba^{-1} = a(b)$ is a path based at $a(1)$. Its homotopy class depends only on the homotopy classes of a and b. This construction defines an isomorphism between $\pi_1(X, a(0))$ and $\pi_1(X, a(1))$.

Proposition 1.6. *Under the hypotheses of (1.5), there exists, up to unique isomorphism, exactly one locally constant sheaf of groups $\Pi_1(X)$ on X (*the fundamental groupoid*), endowed, for all $x_0 \in X$, with an isomorphism*

$$\Pi_1(X)_{x_0} \simeq \pi_1(X, x_0) \tag{1.6.1}$$

and such that, for every path $a\colon [0,1] \to X$, the isomorphism in (1.5) between $\pi_1(X, a(0))$ and $\pi_1(X, a(1))$ can be identified, via (1.6.1)*, with the isomorphism in (1.2) between $\Pi_1(X)_{a(0)}$ and $\Pi_1(X)_{a(1)}$.*

If X is connected, with base point x_0, then the sheaf $\Pi_1(X)$ corresponds, via the equivalence in (1.3), to the group $\pi_1(X, x_0)$ endowed with its action over itself by inner automorphisms.

Proposition 1.7. *If $\mathscr{F}$ is a locally constant sheaf on X, then there exists exactly one action (said to be* canonical*) of $\Pi_1(X)$ on $\mathscr{F}$ that, at each $x_0 \in X$, induces the action from (1.2) of $\pi_1(X, x_0)$ on $\mathscr{F}$.*

I.2 Integrable connections and local systems

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2.1. Let X be an analytic space (0.1). We define a (holomorphic) *vector bundle* on X to be a locally free sheaf of modules that is of finite type over the structure sheaf $\mathscr{O}$ of X. If $\mathscr{V}$ is a vector bundle on X, and x a point of X, then we denote by $\mathscr{V}_{(x)}$ the free $\mathscr{O}_{(x)}$-module of finite type of germs of sections of $\mathscr{V}$. If $\mathfrak{m}_x$ is the maximal ideal of $\mathscr{O}_{(x)}$, then we define the *fibre at x of the vector bundle $\mathscr{V}$* to be the **!TODO!** of finite rank

$$\mathscr{V}_x = \mathscr{V}_{(x)} \otimes_{\mathscr{O}_{(x)}} \mathscr{O}_{(x)}/\mathfrak{m}_x. \tag{2.1.1}$$

If $f\colon X \to Y$ is a morphism of analytic spaces, then the *inverse image* of a vector bundle $\mathscr{V}$ on Y is the vector bundle $f^*\mathscr{V}$ on X given by the inverse image of $\mathscr{V}$ as a coherent module: if $f^\bullet\mathscr{V}$ is the sheaf-theoretic inverse image of $\mathscr{V}$, then

$$f^*\mathscr{V} \simeq \mathscr{O}_X \otimes_{f^*\mathscr{O}_Y} f^\bullet\mathscr{V} \tag{2.1.2}$$

In particular, if $x\colon P \to X$ is the morphism from the point space P to X defined by a point x of X, then

$$\mathscr{V}_x \simeq x^*\mathscr{V}. \tag{2.1.3}$$

2.2. Let X be a complex-analytic manifold (0.7) and $\mathscr{V}$ a vector bundle on X. The elders would have defined a (holomorphic) connection on $\mathscr{V}$ as the data, for every pair of points (x, y) that are first order infinitesimal neighbours in X, of an isomorphism $\gamma_{y,x}\colon \mathscr{V}_x \to \mathscr{V}_y$ that depends holomorphically on (x, y) and is such that $\gamma_{x,x} = \mathrm{Id}$.

Suitably interpreted, this "definition" coincides with the currently fashionable definition ((2.2.4)) given below (which we not be use in the rest of the section).

It suffices to understand "point" to mean "point with values in any analytic space": *p. 6*

2.2.1. *A point in an analytic space X with values in an analytic space S* is a morphism from S to X.

2.2.2. If Y is a subspace of X, then the *nth infinitesimal neighbourhood* of Y in X is the subspace of X defined locally by the $(n+1)$th power of the ideal of $\mathscr{O}_X$ that defines Y.

2.2.3. Two points $x, y \in X$ with values in S are said to be *first order infinitesimal neighbours* if the map $(x, y)\colon S \to X \times X$ that they define factors through the first order infinitesimal neighbourhood of the diagonal of $X \times X$.

2.2.4. If X is a complex-analytic manifold and $\mathscr{V}$ is a vector bundle on X, then a *(holomorphic) connection* γ on $\mathscr{V}$ consists of the following data:

for every pair (x, y) of points of X with values in an arbitrary analytic space S, with x and y first order infinitesimal neighbours, an isomorphism $\gamma_{x,y}\colon x^*\mathscr{V} \to y^*\mathscr{V}$; this data is subject to the conditions:

(i) (functoriality) For any $f\colon T \to S$ and any first order infinitesimal neighbours $x, y\colon S \rightrightarrows X$, we have $f^*(\gamma_{y,x}) = \gamma_{yf,xf}$.

(ii) We have $\gamma_{x,x} = \mathrm{Id}$.

2.3. Let X_1 be the first-order infinitesimal neighbourhood of the diagonal X_0 of $X \times X$, and let p_1 and p_2 be the two projections of X_1 to X. By definition, the vector bundle $P^1(\mathscr{V})$ of first-order jets of sections of $\mathscr{V}$ is the bundle $(p_1)_* p_2^* \mathscr{V}$. We denote by j^1 the first-order differential operator that sends each section of $\mathscr{V}$ to its first-order jet:

$$j^1\colon \mathscr{V} \to P^1(\mathscr{V}) \simeq \mathscr{O}_{X_1} \otimes_{\mathscr{O}_X} \mathscr{V}.$$

A connection ((2.2.4)) can be understood as a homomorphism (which is automatically an isomorphism)

$$\gamma = p_1^*\mathscr{V} \to p_2^*\mathscr{V} \tag{2.3.1}$$

which induces the identity over X_0. Since

$$\mathrm{Hom}_{X_1}(p_1^*\mathscr{V}, p_2^*\mathscr{V}) \simeq \mathrm{Hom}(\mathscr{V}, (p_1)_* p_2^*\mathscr{V}),$$

a connection can also be understood as a ($\mathscr{O}$-linear) homomorphism *p. 7*

$$\mathrm{D}\colon \mathscr{V} \to P^1(\mathscr{V}) \tag{2.3.2}$$

such that the obvious composite arrow

$$\mathscr{V} \xrightarrow{\mathrm{D}} P^1(\mathscr{V}) \to \mathscr{V}$$

is the identity. The sections $\mathrm{D}s$ and $j^1(s)$ of $P^1(v)$ thus have the same image in $\mathscr{V}$, and $j^1(s) - \mathrm{D}(s)$ can be identified with a section ∇s of $\Omega^1_X \otimes \mathscr{V} \simeq \mathrm{Ker}(P^1(\mathscr{V}) \to \mathscr{V})$:

$$\nabla\colon \mathscr{V} \to \Omega^1(X) \tag{2.3.3}$$

$$j^1(s) = \mathrm{D}(s) + \nabla s. \tag{2.3.4}$$

In other words, a connection ((2.2.4)), allowing us to compare two neighbouring fibres of $\mathscr{V}$, also allows us to define the differential ∇s of a section of $\mathscr{V}$.

Conversely, equation (2.3.4) allows us to define D, and thus γ, from the covariant derivative ∇. For D to be linear, it is necessary and sufficient for ∇ to satisfy the identity

$$\nabla(fs) = \mathrm{d}f \cdot s + f \cdot \nabla s \tag{2.3.5}$$

Definition (2.2.4) is thus equivalent to the following definition, due to J.L. Koszul.

Definition 2.4. Let $\mathscr{V}$ be a (holomorphic) vector bundle on a complex-analytic manifold X. A ***holomorphic connection*** (or simply, ***connection***) on $\mathscr{V}$ is a $\mathbb{C}$-linear homomorphism

$$\nabla\colon \mathscr{V} \to \Omega^1_X(\mathscr{V}) = \Omega^1_X \otimes_{\mathscr{O}} \mathscr{V}$$

that satisfies the Leibniz identity ((2.3.5)) for local sections f of $\mathscr{O}$ and s of $\mathscr{V}$. We call ∇ the *covariant derivative* defined by the connection.

2.5. If the vector bundle $\mathscr{V}$ is endowed with a connection Γ with covariant derivative ∇, and if w is a holomorphic vector field on X, then we set, for every local section v of $\mathscr{V}$ over an open subset U of X,

$$\nabla_w(v) = \langle \nabla v, w \rangle \in \mathscr{V}(U).$$

We call $\nabla_w\colon \mathscr{V} \to \mathscr{V}$ the ***covariant derivative along the vector field*** w.

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2.6. If ${}_1\Gamma$ and ${}_2\Gamma$ are connections on X, with covariant derivatives ${}_1\nabla$ and ${}_2\nabla$ (respectively), then ${}_2\nabla - {}_1\nabla$ is a $\mathscr{O}$-linear homomorphism from $\mathscr{V}$ to $\Omega^1_X(\mathscr{V})$. Conversely, the sum of ${}_1\nabla$ and such a homomorphism defines a connection on $\mathscr{V}$. Thus connections on $\mathscr{V}$ form a principal homogeneous space (or torsor) on $\underline{\mathrm{Hom}}(\mathscr{V}, \Omega^1_X(\mathscr{V})) \simeq \Omega^1_X(\underline{\mathrm{End}}(\mathscr{V}))$.

2.7. If vector bundles are endowed with connections, then every vector bundle obtained by a "tensor operation" is again endowed with a connection. This is evident with (2.2.4). More precisely, let $\mathscr{V}_1$ and $\mathscr{V}_2$ be vector bundles endowed with connections with covariant derivatives ∇_1 and ∇_2.

2.7.1. We define a connection on $\mathscr{V}_1 \oplus \mathscr{V}_2$ by the formula

$$\nabla_w(v_1 + v_2) = {}_1\nabla_w(v_1) + {}_2\nabla_w(v_2)$$

2.7.2. We define a connection on $\mathscr{V}_1 \otimes \mathscr{V}_2$ by the Leibniz formula

$$\nabla_w(v_1 \otimes v_2) = \nabla_w v_1 \cdot v_2 + v_1 \cdot \nabla_w v_2.$$

2.7.3. We define a connection on $\underline{\mathrm{Hom}}(\mathscr{V}_1, \mathscr{V}_2)$ by the formula

$$(\nabla_w f)(v_1) = {}_2\nabla_2(f(v_1)) - f({}_1\nabla v_1).$$

The canonical connection on $\mathscr{O}$ is the connection for which $\nabla f = \mathrm{d}f$.

Let $\mathscr{V}$ be a vector bundle endowed with a connection.

2.7.4. We define a connection on the dual $\mathscr{V}^\vee$ of $\mathscr{V}$ via (2.7.3) and the defining isomorphism $\mathscr{V}^\vee = \underline{\mathrm{Hom}}(\mathscr{V}, \mathscr{O})$. We have

$$\langle \nabla_w v', v \rangle = \partial_w \langle v', v \rangle - \langle v', \nabla_w v \rangle.$$

We leave it to the reader to verify that these formulas do indeed define connections. For (2.7.2), for example, one must verify that, firstly, the given formula defines a $\mathbb{C}$-bilinear map from $(\mathscr{V}_1 \otimes \mathscr{V}_2)$, which means that the right-hand side $\mathrm{II}(v_1, v_2)$ is $\mathbb{C}$-bilinear and such that $\mathrm{II}(fv_1, v_2) = \mathrm{II}(v_1, fv_2)$; secondly, one must also verify identity (2.3.5).

2.8. An $\mathscr{O}$-homomorphism f between vector bundles $\mathscr{V}_1$ and $\mathscr{V}_2$ endowed with connections is said to be ***compatible with the connections*** if *p. 9*

$${}_2\nabla \cdot f = f \cdot {}_1\nabla.$$

By (2.7.3), this reduces to saying that $\nabla f = 0$, if f is thought of as a section of $\underline{\mathrm{Hom}}(\mathscr{V}_1, \mathscr{V}_2)$. For example, by (2.7.3), the canonical map

$$\mathrm{Hom}(\mathscr{V}_1, \mathscr{V}_2) \otimes \mathscr{V}_1 \to \mathscr{V}_2$$

is compatible with the connections.

2.9. A local section v of $\mathscr{V}$ is said to be ***horizontal*** if $\nabla v = 0$. If f is a homomorphism between bundles $\mathscr{V}_1$ and $\mathscr{V}_2$ endowed with connections, then it is equivalent to say either that f is horizontal, or that f is compatible with the connections (2.8).

2.10. Let $\mathscr{V}$ be a holomorphic vector bundle on X. Define $\Omega_X^p = \wedge^p \Omega_X^1$ and $\Omega_X^p(\mathscr{V}) = \Omega_X^p \otimes_{\mathscr{O}} \mathscr{V}$ (the sheaf of ***exterior differential p-forms with values in*** $\mathscr{V}$). Suppose that $\mathscr{V}$ is endowed with a holomorphic connection. We then define $\mathbb{C}$-linear morphisms

$$\nabla\colon \Omega_X^p(\mathscr{V}) \to \Omega_X^{p+1}(\mathscr{V}) \tag{2.10.1}$$

characterised by the following formula:

$$\nabla(\alpha, v) = \mathrm{d}\alpha \cdot v + (-1)^p \alpha \wedge \nabla v, \tag{2.10.2}$$

where α is any local section of Ω^p, v is any local section of $\mathscr{V}$, and d is the exterior differential. To prove that the right-hand side $\mathrm{II}(\alpha, v)$ of (2.10.2) defines a homomorphism (2.10.1), it suffices to show that $\mathrm{II}(\alpha, v)$ is $\mathbb{C}$-bilinear and satisfies

$$\mathrm{II}(f\alpha, v) = \mathrm{II}(\alpha, fv).$$

But we have that

$$\begin{aligned}\mathrm{II}(f\alpha, v) &= \mathrm{d}(f\alpha)v + (-1)^p f\alpha \wedge \nabla v \\ &= \mathrm{d}\alpha \cdot fv + \mathrm{d}f \wedge \alpha v + (-1)^p f\alpha \wedge \nabla v \\ &= \mathrm{d}\alpha \cdot fv + (-1)^p \alpha \wedge (f\nabla v + \mathrm{d}f \cdot v) \\ &= \mathrm{II}(\alpha, fv).\end{aligned}$$

Let $\mathscr{V}_1$ and $\mathscr{V}_2$ be vector bundles endowed with connections, and let $\mathscr{V}$ be their tensor product (2.7.2). We denote by $\wedge$ the evident maps

$$\wedge\colon \Omega^p(\mathscr{V}_1) \otimes \Omega^1(\mathscr{V}_2) \to \Omega^{p+q}(\mathscr{V})$$

such that, for any local section α (resp. β, resp. v_1, resp. v_2) of Ω^p (resp. Ω^q, resp. $\mathscr{V}_1$, resp. $\mathscr{V}_2$), we have that $(\alpha \otimes v_1) \wedge (\beta \otimes v_2) = (\alpha \wedge \beta) \otimes (v_1 \otimes v_2)$. If ν_1 (resp. ν_2) is any local section of $\Omega^p(\mathscr{V}_1)$ (resp. $\Omega^q(\mathscr{V}_2)$), then p. 10

$$\nabla(\nu_1 \wedge \nu_2) = \nu_1 \wedge \nu_2 + (-1)^p \nu_1 \wedge \nu_2. \tag{2.10.3}$$

Indeed, if $\nu_1 = \alpha v_1$ and $\nu_2 = \beta v_2$, then

$$\begin{aligned}\nabla(\nu_1 \wedge \nu_2) &= \nabla(\alpha \wedge \beta \otimes v_1 \otimes v_2) \\ &= \mathrm{d}(\alpha \wedge \beta) v_1 \otimes v_2 + (-1)^{p+q} \alpha \wedge \beta \wedge \nabla(v_1 \otimes v_2) \\ &= \mathrm{d}\alpha \wedge \beta v_1 \otimes v_2 + (-1)^p \alpha \wedge \mathrm{d}\beta v_1 \otimes v_2 \\ &\quad + (-1)^{p+q} \alpha \wedge \beta \wedge \nabla v_1 \otimes v_2 + (-1)^{p+q} \alpha \wedge \beta v_1 \wedge \nabla v_2 \\ &= \mathrm{d}\alpha v_1 \wedge \nu_2 + (-1)^p \nu_1 \wedge \mathrm{d}\beta v_2 + (-1)^p \alpha \wedge \nabla v_1 \wedge \nu_2 \\ &\quad + (-1)^{p+q} \nu_2 \wedge \beta \wedge \nabla v_2 \\ &= (\mathrm{d}\alpha v_1 + (-1)^p \alpha \wedge \nabla v_1) \wedge \nu_2 + (-1)^p \nu_1 \wedge (\mathrm{d}\beta v_2 + (-1)^q \beta \wedge \nabla v_2) \\ &= \nabla \nu_1 \wedge \nu_2 + (-1)^p \nu_1 \wedge \nabla \nu_2.\end{aligned}$$

Let $\mathscr{V}$ be a vector bundle endowed with a connection. If we apply the above formula to $\mathscr{O}$ and $\mathscr{V}$, then, for any local section α (resp. ν) of Ω^p (resp. $\Omega^q(\mathscr{V})$), we have that

$$\nabla(\alpha \wedge \nu) = \mathrm{d}\alpha \wedge \nu + (-1)^p \alpha \wedge \nabla \nu. \tag{2.10.4}$$

Iterating this formula gives

$$\begin{aligned}\nabla\nabla(\alpha \wedge \nu) &= \nabla(\mathrm{d}\alpha \wedge \nu + (-1)^p \alpha \wedge \nabla \nu) \\ &= \mathrm{d}\alpha \wedge \nu + (-1)^{p+1} \mathrm{d}\alpha \wedge \nabla \nu + (-1)^p \mathrm{d}\alpha \wedge \nabla \nu + \alpha \wedge \nabla\nabla \nu \\ &= \alpha \wedge \nabla\nabla \nu.\end{aligned} \tag{2.10.5}$$

Definition 2.11. Under the hypotheses of (2.10), the *curvature* R of the given connection on $\mathscr{V}$ is the composite homomorphism

$$R\colon \mathscr{V} \to \Omega^2_X(\mathscr{V})$$

considered as a section of $\mathrm{Hom}(\mathscr{V}, \Omega^2_X(\mathscr{V})) \simeq \Omega^2_X(\mathrm{End}(\mathscr{V}))$.

2.12. Taking $q = 0$ in (2.10.4) gives

$$\nabla\nabla(\alpha v) = \alpha \wedge R(v), \tag{2.12.1}$$

which we write as *p. 11*

$$\nabla\nabla(\nu) = R \wedge \nu \qquad \text{(the } \textit{Ricci identity}\text{)}. \tag{2.12.2}$$

We endow $\underline{\mathrm{End}}(\mathscr{V})$ with the connection given in (2.7.3). The equation $\nabla(\nabla\nabla) = (\nabla\nabla)\nabla$ can be written as $\nabla(R \wedge \nu) = R \wedge \nabla\nu$. By (2.7.3), we have that $\nabla R \wedge \nu = \nabla(R \wedge \nu) - R \wedge \nabla\nu$, so that

$$\nabla R = 0 \qquad \text{(the } \textit{Bianchi identity}\text{)}. \tag{2.12.3}$$

2.13. If α is an exterior differential p-form, then we know that

$$\begin{aligned}\langle \mathrm{d}\alpha, X_0 \wedge \ldots \wedge X_p\rangle = &\sum_i (-1)^i j_{X_i} \langle \alpha, X_0 \wedge \ldots \wedge \widehat{X_i} \wedge \ldots \wedge X_p\rangle \\ &+ \sum_{i<j})(-1)^{i+j} \langle \alpha, [X_i, X_j] \wedge X_0 \wedge \ldots \wedge \widehat{X_i} \wedge \ldots \wedge \widehat{X_j} \wedge \ldots \wedge X_p\rangle.\end{aligned}$$

From this formula, and from (2.10.2), we see that, for any local section ν of $\Omega_X^p(\mathscr{V})$, and holomorphic vector fields $X_0, \ldots, X_p$,

$$\begin{aligned}\langle \nabla\nu, X_0 \wedge \ldots \wedge X_p\rangle = &\sum_i (-1)^i \nabla_{X_i} \langle \nu, X_0 \wedge \ldots \wedge \widehat{X_i} \wedge \ldots \wedge X_p\rangle \\ &+ \sum_{i<j})(-1)^{i+j} \langle \nu, [X_i, X_j] \wedge X_0 \wedge \ldots \wedge \widehat{X_i} \wedge \ldots \wedge \widehat{X_j} \wedge \ldots \wedge X_p\rangle.\end{aligned} \tag{2.13.1}$$

In particular, for any local section v of $\mathscr{V}$, we have that

$$\langle \nabla\nabla v, X_1 \wedge X_2\rangle = \nabla_{X_1}\langle \nabla v, X_2\rangle - \nabla_{X_2}\langle v, X_1\rangle - \langle \nabla v, [X_1, X_2]\rangle.$$

That is,

$$R(X_1, X_2)(v) = \nabla_{X_1}\nabla_{X_2} v - \nabla_{X_2}\nabla_{X_1} v - \nabla_{[X_1,X_2]} v. \tag{2.13.2}$$

Definition 2.14. A connection is said to be ***integrable*** if its curvature is zero, i.e. (2.13.2) if the following holds identically:

$$\nabla_{[X,Y]} = [\nabla_X, \nabla_Y].$$

If $\dim(X) \leqslant 1$, then every connection is integrable.

If Γ is an integrable connection on $\mathscr{V}$, then the morphism ∇ of (2.10.1) satisfy $\nabla\nabla = 0$, and so the $\Omega^p(\mathscr{V})$ give a differential complex $\Omega^\bullet(\mathscr{V})$.

Definition 2.15. Under the above hypotheses, the complex $\Omega^\bullet(\mathscr{V})$ is called the ***holomorphic de Rham complex*** with values in $\mathscr{V}$.

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The results (2.16) to (2.19) that follow will be proven in a more general setting in **??**.

Proposition 2.16. *Let V be a local complex system on a complex-analytic variety X (0.6), and let $\mathscr{V} = \mathscr{O} \otimes_{\mathbb{C}} V$.*

(i) *There exists, on the vector bundle $\mathscr{V}$, exactly one connection (said to be* canonical*) whose horizontal sections are the local sections of the subsheaf V of $\mathscr{V}$.*

(ii) *The canonical connection on $\mathscr{V}$ is integrable.*

(iii) *For any local section f (resp. v) of $\mathscr{O}$ (resp. V),*

$$\nabla(fv) = \mathrm{d}f \cdot v. \tag{2.16.1}$$

Proof. If ∇ satisfies (i), then (2.16.1) is a particular case of (2.3.5). Conversely, the right-hand side $\mathrm{II}(f,v)$ of (2.16.1) is $\mathbb{C}$-bilinear, and thus extends uniquely to a $\mathbb{C}$-linear map $\nabla\colon \mathscr{V} \to \Omega^1(\mathscr{V})$, which we can show defines a connection. Claim (ii) is local on X, which allows us to reduce to the case where $V = \underline{\mathbb{C}}$. Then $\mathscr{V} = \mathscr{O}$, $\nabla = \mathrm{d}$, and $\nabla_{[X,Y]} = [\nabla_X, \nabla_Y]$ by the definition of $[X,Y]$. □

It is well known that:

Theorem 2.17. *Let X be a complex-analytic variety. Then the following functors are equivalences of categories, quasi-inverse to one another, between the category of complex local systems on X and the category of holomorphic vector bundles with on X with integrable connections (with the morphisms being the horizontal morphisms of vector bundles):*

a) *the complex local system V is sent to $\mathscr{V} = \mathscr{O} \otimes V$ endowed with its canonical connection;*

b) *the holomorphic vector bundle $\mathscr{V}$ endowed with its integrable connection is sent to the subsheaf V of $\mathscr{V}$ consisting of horizontal sections (i.e. those v such that $\nabla v = 0$).*

These equivalences are compatible with taking the tensor product, the internal Hom, and the dual; to the unit complex local system $\underline{\mathbb{C}}$ corresponds the bundle $\mathscr{O}$ endowed with the connection ∇ such that $\nabla f = \mathrm{d}f$.

Definition (2.10.2) implies the following:

Proposition 2.18. *If V is a complex local system on X, and if $\mathscr{V} = \mathscr{O} \otimes_{\mathbb{C}} V$, then the system of isomorphisms* p. 13

$$\Omega_X^p \otimes_{\mathbb{C}} V \simeq \Omega_X^p \otimes_{\mathscr{O}} \mathscr{O} \otimes_{\mathbb{C}} V \simeq \Omega_X^p \otimes_{\mathscr{O}} \mathscr{V}$$

is an isomorphism of complexes

$$\Omega_X^\bullet \otimes_{\mathbb{C}} V \to \Omega_X^\bullet(\mathscr{V}).$$

From this, the holomorphic Poincaré lemma gives the following:

Proposition 2.19. *Under the hypotheses of (2.16), the complex $\Omega_X^\bullet(\mathscr{V})$ is a resolution of the sheaf $\mathscr{V}$.*

2.20. Variants.

2.20.1. If X is a differentiable manifold, and we consider C^∞ connections on C^∞ vector bundles, then all of the above results still hold true, mutatis mutandis. We will not use this fact.

2.20.2. Theorem (2.17) makes essential use of the non-singularity of X; it is thus unimportant to note that this hypothesis has not been used in an essential way before (2.17)

2.20.3. The definition (2.4) of a connection and the definition (2.11) of an integrable connection are formal enough that we can transport them to the category of schemes, or in relative settings:

Definition 2.21. (i) Let